On the spectrum of frequently hypercyclic operators

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Abstract

A bounded linear operator T on a Banach space X is called frequently hypercyclic if there exists $x \in X$ such that the lower density of the set $\{n \in \mathbb{N} : T^n x \in U\}$ is positive for any non-empty open subset U of X. Bayart and Grivaux have raised a question whether there is a frequently hypercyclic operator on any separable infinite dimensional Banach space. We prove that the spectrum of a frequently hypercyclic operator has no isolated points. It follows that there are no frequently hypercyclic operators on all complex and on some real hereditarily indecomposable Banach spaces, which provides a negative answer to the above question.

1 Introduction

All vector spaces in this article are assumed to be over the field $\mathbb{K}$ being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+$ is the set of non-negative integers and $\mathbb{N}$ is the set of positive integers. For a Banach space X, $L(X)$ stands for the algebra of bounded linear operators on X and X^* is the space of continuous linear functionals on X.

Recall that a continuous linear operator T on a topological vector space X is called *hypercyclic* if there is $x \in X$ such that the orbit $\{T^n x : x \in \mathbb{Z}_+\}$ is dense in X. We refer to surveys [17, 18] for additional information on hypercyclicity. There are several stronger versions of the above property of linear operators like hereditarily hypercyclic operators, topologically mixing operators and operators satisfying the Kitai Criterion. Recently Bayart and Grivaux [5] have introduced the concept of frequently hypercyclic operators. Recall that the *upper* and *lower* densities of a set $A \subset \mathbb{Z}_+$ are defined by the formula

$$\underline{d}(A) = \varliminf_{n \to \infty} \frac{|\{m \in A : m \leqslant n\}|}{n}, \qquad \overline{d}(A) = \varlimsup_{n \to \infty} \frac{|\{m \in A : m \leqslant n\}|}{n},$$

where $|B|$ stands for the number of elements of a finite set B.

Definition 1. Let T be a continuous linear operator on a topological vector space X. Then $x \in X$ is called a *frequently hypercyclic* vector for T if the lower density of the set $\{n \in \mathbb{N} : T^n x \in U\}$ is positive for any non-empty open subset U of X. An operator T is called *frequently hypercyclic* if it has a frequently hypercyclic vector.

We also say that $x \in X$ is an $\mathfrak{u}$-*frequently hypercyclic* vector for T if the upper density of the set $\{n \in \mathbb{N} : T^n x \in U\}$ is positive for any non-empty open subset U of X. An operator T is called $\mathfrak{u}$-*frequently hypercyclic* if it has an $\mathfrak{u}$-frequently hypercyclic vector. We denote the set of $\mathfrak{u}$-frequently hypercyclic vectors for T by the symbol $\mathfrak{u}(T)$.

Clearly frequent hypercyclicity implies $\mathfrak{U}$-frequent hypercyclicity, which, in turn, implies hypercyclicity. Bayart and Grivaux [5] have raised the following question.

Question 1. Does every separable infinite dimensional Banach space support a frequently hypercyclic operator?

This question is partially motivated by the following chain of results. Ansari [1] and Bernal-Gonzáles [6] showed independently that for any separable infinite dimensional Banach space X there is a hypercyclic operator $T \in L(X)$. Grivaux [15] proved that there is a mixing operator on any separable infinite dimensional Banach space. Finally, in [25] it is shown that on any separable infinite dimensional Banach space, there is a bounded linear operator, satisfying the Kitai Criterion. In this article we show that unlike for other strong forms of hypercyclicity, the answer to the above question is negative.

Theorem 1.1. *Let T be a bounded linear operator on a separable infinite dimensional Banach space X such that $\lim_{n \to \infty} \|T^n\|^{1/n} = 0$. Then $I + T$ is not $\mathfrak{U}$-frequently hypercyclic. In particular, $I + T$ is not frequently hypercyclic.*

In the case $\mathbb{K} = \mathbb{C}$, we use the above theorem to prove the following result.

Theorem 1.2. *Let X be a complex Banach space and $T \in L(X)$ be frequently hypercyclic. Then the spectrum $\sigma(T)$ of T has no isolated points.*

This result shows that the class of frequently hypercyclic operators is significantly smaller than the class of hypercyclic operators. For instance, in [16] it is demonstrated that if X is a separable Banach space and M is the operator norm closure of the set of finite rank nilpotent operators, then the set $\{T \in M : I + T \text{ is hypercyclic}\}$ is a dense G_δ subset of M. Thus, in a sense, a 'generic' operator with one-point spectrum $\{1\}$ is hypercyclic.

We use Theorem 1.1 to obtain a negative answer to Question 1. It turns out that there are no frequently hypercyclic operators on Banach spaces with few operators. Recall that a bounded linear operator T on a Banach space X is called *strictly singular* if the restriction of T to any closed infinite dimensional subspace of X is not an isomorphism onto its image. As is well-known [24], the set $\mathcal{S}(X)$ of strictly singular operators is an ideal in the algebra $L(X)$.

By $\Phi_{\mathbb{K}}$ we denote the class of infinite dimensional Banach spaces X over the field $\mathbb{K}$ for which $L(X)/\mathcal{S}(X)$ is isomorphic to $\mathbb{K}$. It is straightforward to see that $X \in \Phi_{\mathbb{K}}$ if and only if any $T \in L(X)$ has the shape $T = zI + S$ for $z \in \mathbb{K}$ and $S \in \mathcal{S}(X)$.

Theorem 1.3. *Let $X \in \Phi_{\mathbb{K}}$. Then there is no bounded $\mathfrak{U}$-frequently hypercyclic operator on X. In particular, X does not support a frequently hypercyclic operator.*

The above theorem gives a negative answer to Question 1 provided there are separable Banach spaces in classes $\Phi_{\mathbb{C}}$ and $\Phi_{\mathbb{R}}$. Recall that an infinite dimensional Banach X space is called *hereditarily indecomposable* or an *HI space* if for any infinite dimensional closed linear subspaces L and M of X satisfying $L \cap M = \{0\}$ their sum $L + M$ is not closed. Separable HI Banach spaces were first constructed by Gowers and Maurey [14], see also the survey [4]. One of the most interesting features of HI spaces X is [14, 12] that any complex HI space belongs to $\Phi_{\mathbb{C}}$. It is also worth noting that there are separable Banach spaces in $\Phi_{\mathbb{C}}$ that are not HI [3]. Anyway, we have the following corollary.

Corollary 1.4. *Let X be a separable complex HI space. Then there is no bounded $\mathfrak{U}$-frequently hypercyclic operator on X. In particular, X does not support a frequently hypercyclic operator.*

The case of real HI spaces is slightly more complicated. Namely, Ferenczi [12, 13] demonstrated that if X is a real HI space, then $L(X)/\mathcal{S}(X)$ is isomorphic to either $\mathbb{R}$, or $\mathbb{C}$ or the quaternion

division ring $\mathbb{H}$. Moreover, in [13] it is shown that all three possibilities do occur on the level of separable real HI spaces and is observed that the original real separable HI space X of Gowers and Maurey [14] satisfies $L(X)/\mathcal{S}(X) \simeq \mathbb{R}$ and therefore belongs to $\Phi_{\mathbb{R}}$. This remark together with Theorem 1.3 gives us the following corollary.

Corollary 1.5. *There is an infinite dimensional separable real Banach space X, which does not support a $\mathfrak{U}$-frequently hypercyclic operator. In particular, X does not support a frequently hypercyclic operator.*

2 Proof of Theorem 1.1

Theorem 2.1. *Let T be a continuous linear operator on a real topological vector space X and $x \in X$ be such that there exists $f \in X^* \setminus \{0\}$ satisfying*

$$\lim_{n \to \infty} |f((T-I)^n x)|^{1/n} = 0 \quad \text{for each} \quad x \in X. \tag{2.1}$$

Then x is not a $\mathfrak{U}$-frequently hypercyclic vector for T.

Proof. Assume that $x \in \mathfrak{U}(T)$. First, observe, that we can find $u \in X$ such that $f(u) < 0$ and $f(Tu) > 0$. Indeed, otherwise, using continuity of f and T, we see that $f(Tu) \leqslant 0$ if $f(u) \leqslant 0$. Since $f(-y) = -f(y)$ for any $y \in X$, we also have $f(Tu) \geqslant 0$ if $f(u) \geqslant 0$. Hence $T(H) \subseteq H$, where $H = \ker f$. Thus, T has a closed invariant subspace of codimension 1. On the other hand, in [26] it is proven that any hypercyclic operator on any real or complex topological vector space has no closed invariant subspaces of finite codimension. This contradiction shows that there is $u \in X$ such that $f(u) < 0$ and $f(Tu) > 0$. Since f and T are continuous, we can choose an open neighborhood U of u such that

$$f(v) < 0 \quad \text{and} \quad f(Tv) > 0 \quad \text{for any } v \in U. \tag{2.2}$$

Since $x \in \mathfrak{U}(T)$, there exists a subset A of $\mathbb{Z}_+$ of positive upper density such that

$$f(T^n x) \in U \text{ for any } n \in A. \tag{2.3}$$

Consider now the series

$$F(z) = f(x) + \sum_{n=1}^{\infty} \frac{f((T-I)^n x)}{n!} z(z-1) \ldots (z-n+1), \quad z \in \mathbb{C}. \tag{2.4}$$

Condition (2.1) implies that the series in (2.4) is uniformly convergent on any compact subset of $\mathbb{C}$ and therefore defines an entire function F. Moreover, F has exponential type 0, that is,

$$\lim_{R \to \infty} \frac{\ln M_F(R)}{R} = 0, \quad \text{where} \quad M_F(R) = \max_{|z| \leqslant R} |F(z)|. \tag{2.5}$$

Indeed, by (2.1), for any $\varepsilon \in (0,1)$, there is $m \in \mathbb{N}$ such that $|f((T-I)^n x)| < \varepsilon^n$ for $n > m$. Then

$$M_F(R) \leqslant |f(x)| + \sum_{n=1}^{\infty} \frac{|f((T-I)^n x)|}{n!} R(R+1) \ldots (R+n-1) \leqslant$$

$$\leqslant 1 + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} R(R+1) \ldots (R+n-1) + O(R^m) = (1-\varepsilon)^{-R} + O(R^m) =$$

$$= e^{a(\varepsilon)R} + O(R^m) = O(e^{a(\varepsilon)R}) \quad \text{as } R \to \infty, \text{ where } a(\varepsilon) = -\ln(1-\varepsilon).$$

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Since $a(\varepsilon) \to 0$ as $\varepsilon \to 0$, (2.5) follows. From the definition of F it also easily follows that

$$F(k) = f(T^k x) \text{ for any } k \in \mathbb{Z}_+. \tag{2.6}$$

Using (2.6), (2.3) and (2.2), we see that $F(n) < 0$ and $F(n+1) > 0$ for any $n \in A$. Since F is real on the real axis, for any $n \in A$, there is $t_n \in (n, n+1)$ such that $F(t_n) = 0$. Let now $N_F(R)$ be the number of zeros of F in the set $\{z \in \mathbb{C} : 0 < |z| < R\}$. The last observation yields $N_F(n+1) \geqslant k_n$ for any $n \in \mathbb{Z}_+$, where $k_n = |\{m \in A : m \leqslant n\}|$ is the counting function of the set A. On the other hand, by a corollary of Jensen's Theorem, see [23, p. 15], we have

$$N_F(R) \leqslant \ln M_F(eR) + O(\ln R) \quad \text{as } R \to \infty. \tag{2.7}$$

Hence

$$\varlimsup_{n\to\infty} \frac{\ln M_F(e(n+1))}{n} \geqslant \varlimsup_{n\to\infty} \frac{N_F(n+1)}{n} \geqslant \varlimsup_{n\to\infty} \frac{k_n}{n} = \overline{d}(A) > 0,$$

which contradicts (2.5). $\qquad\square$

Corollary 2.2. *Let X be a Banach space and $T \in L(X)$ be such that there exists $f \in X^* \setminus \{0\}$ for which $\lim\limits_{n\to\infty} \|T^{*n} f\|^{1/n} = 0$. Then $I + T$ is not $\mathfrak{u}$-frequently hypercyclic.*

Proof. If $\mathbb{K} = \mathbb{R}$, we just apply Theorem 2.1 to the operator $I + T$ to see that the latter has no $\mathfrak{u}$-frequently hypercyclic vectors. If $\mathbb{K} = \mathbb{C}$, we consider X as a real Banach space, T as an $\mathbb{R}$-linear operator and apply Theorem 2.1 to the operator $I + T$ and the functional $\operatorname{Re} f$. $\qquad\square$

Theorem 1.1 follows immediately from Corollary 2.2 since $\|T^{*n} f\| \leqslant \|f\| \|T^{*n}\| = \|f\| \|T^n\|$ for any $f \in X^*$ and any $n \in \mathbb{Z}_+$.

3 Rotations and powers of $\mathfrak{u}$-frequently hypercyclic operators

In order to reduce Theorem 1.3 and Theorem 1.2 to Theorem 1.1, we need to show that the statement of Theorem 1.1 remains true if we replace $I + T$ be $zI + T$ for any $z \in \mathbb{K}$ with $|z| = 1$. In order to do so, we need to demonstrate that the class of $\mathfrak{u}$-frequently hypercyclic operators is stable under multiplication by scalars z with $|z| = 1$.

For hypercyclic operators acting on complex Banach spaces, this kind of stability was proved by León-Saavedra and Müller [22]. In the case $\mathbb{K} = \mathbb{R}$ the fact that T is hypercyclic if and only if $-T$ is hypercyclic follows from the result of Ansari [2], who showed that hypercyclicity of T implies hypercyclicity of its powers. In particular T^2 is hypercyclic if T is. Since $(-T)^2 = T^2$, the latter observation shows that T is hypercyclic if and only if $-T$ is. We are going to prove the same stability results for $\mathfrak{u}$-frequently hypercyclic operators. Concerning the powers, we shall use a result of Bourdon and Feldman [10], who demonstrated that an orbit of a continuous linear operator on a locally convex topological vector space is either everywhere dense or nowhere dense. The only obstacle that did not allow the same construction to work for arbitrary topological vector spaces was that the proof of the density of the range of $p(T)$ for any non-zero polynomial p and a hypercyclic operator T used the duality argument. This obstacle was removed by Wengenroth [26], which allowed him to extend the Bourdon and Feldman theorem to arbitrary topological vector spaces, see also the survey [18]. Throughout this section X is a topological vector space and $T \in L(X)$.

Theorem A. *Let $x \in X$ be such that $F_x^T = \overline{\{T^n x : n \in \mathbb{Z}_+\}}$ has nonempty interior. Then $F_x^T = X$.*

Although the above mentioned result of León-Saavedra and Müller is formulated and proved for operators acting on Banach spaces, the same proof with minor modifications (just replace convergent sequences by convergent nets and use the result of Wengenroth to ensure density of the ranges of polynomials of T instead of the duality argument) gives the following extension of their result.

Theorem B. *Let $z \in \mathbb{K}$, $|z| = 1$. Then the sets of hypercyclic vectors for T and zT coincide.*

For shortness, we denote by Λ the set of all subsets of $\mathbb{Z}_+$ of positive upper density. We also fix a base $\mathcal{U}$ of open neighborhoods of 0 in X.

Definition 2. For $x \in X$ we denote

$$M_x^T = \{y \in X : \{n \in \mathbb{Z}_+ : y - T^n x \in U\} \in \Lambda \ \text{ for any } U \in \mathcal{U}\}. \tag{3.1}$$

Clearly, $x \in \mathfrak{U}(T)$ if and only if $M_x^T = X$.

If additionally X is complex, we denote

$$N_x^T = \{y \in X : \{n \in \mathbb{Z}_+ : C_n(T, x) \cap (y + U) \neq \varnothing\} \in \Lambda \ \text{ for any } U \in \mathcal{U}\}, \tag{3.2}$$
$$\text{where } \ C_n(T, x) = \{zT^n x : z \in \mathbb{T}\}.$$

We start by establishing few straightforward properties of the sets M_x^T and N_x^T defined in (3.1) and (3.2). The following lemma is an elementary exercise. We leave the proof to the reader.

Lemma 3.1. *For any $x \in X$, M_x^T is closed in X, $T(M_x^T) \subseteq M_x^T$, $M_{Tx}^T = M_x^T$ and $M_{zx}^T = zM_x^T$ for each $z \in \mathbb{K}$. If X is complex, then for any $x \in X$, N_x^T is closed in X, $T(N_x^T) \subseteq N_x^T$, $N_{Tx}^T = N_x^T$ and $N_{zx}^T = N_x^{zT} = N_x^T$ for any $z \in \mathbb{T}$.*

Lemma 3.2. *If $u, x, y \in X$ are such that $y \in M_x^T$ and $x \in M_u^T$, then $y \in M_u^T$.*

Proof. Let $U \in \mathcal{U}$. Pick $V \in \mathcal{U}$ such that $V + V \subset U$. Since $y \in M_x^T$, we can find $m \in \mathbb{Z}_+$ such that $y - T^m x \in V$. Since $x \in M_u^T$, we have $A = \{k \in \mathbb{Z}_+ : x - T^k u \in (T^m)^{-1}(V)\} \in \Lambda$. Then for any $k \in A$, we have

$$y - T^{k+m} u = y - T^m x + T^m(x - T^k u) \in V + T^m((T^m)^{-1}(V)) \subseteq V + V \subseteq U.$$

Thus $y - T^n u \in U$ for any $n \in m + A$. Since $m + A \in \Lambda$, we have $y \in M_u^T$. $\qquad\square$

Lemma 3.3. *If X is complex and $x \in N_x^T$, then $x \in M_x^T$.*

Proof. Clearly the set $F = \{\mu \in \mathbb{T} : \mu x \in M_x^T\}$ is closed since M_x^T is. Let us verify that F is non-empty. For each $U \in \mathcal{U}$ consider

$$F(U) = \{z \in \mathbb{T} : \{n \in \mathbb{Z}_+ : zx - T^n x \in U\} \in \Lambda\}$$

and $\overline{F}(U)$ be the closure of $F(U)$ in $\mathbb{T}$. Clearly $F = \bigcap\{F(U) : U \in \mathcal{U}\}$. Let us show that the sets $F(U)$ are non-empty. Fix $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ and $\varepsilon > 0$ such that $V + D_\varepsilon x \subseteq U$, where $D_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$. Since $x \in N_x^T$, there exist $A \in \Lambda$ and a sequence $\{z_m\}_{m \in A}$ of elements of $\mathbb{T}$ such that $z_m x - T^m x \in V$ for any $m \in A$. Let $k \in \mathbb{N}$ be such that $k > \varepsilon^{-1}$ and let $w_j = e^{2\pi ij/k}$ for $0 \leqslant j \leqslant k - 1$. Then for any $m \in A$ we can pick $\nu(m) \in \{0, \ldots, k-1\}$ such that $|z_m - w_{\nu(m)}| < \varepsilon$. Clearly A is the union of $A_j = \{m \in A : \nu(m) = j\}$ for $0 \leqslant j \leqslant k - 1$. Since the union of A_j has positive upper density, there exists $j \in \{0, \ldots, k-1\}$ such that $A_j \in \Lambda$. For each $m \in A_j$, we have

$$w_j x - T^m x = z_m x - T^m x + (w_j - z_m)x \in V + D_\varepsilon x \subset U.$$

Hence $w_j \in F(U)$ and $F(U)$ is non-empty. From the definition of the set $F(U)$, we also have that $\overline{F}(V) \subset J_\varepsilon F(V) \subseteq F(U)$, where $J_\varepsilon = \{e^{it} : -\varepsilon \leqslant t \leqslant \varepsilon\}$. Thus for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\overline{F}(V) \subset F(U)$. It follows that

$$F = \bigcap_{U \in \mathcal{U}} F(U) = \bigcap_{U \in \mathcal{U}} \overline{F}(U). \tag{3.3}$$

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Finally, for any $U_1, \ldots, U_n \in \mathcal{U}$, we have

$$\varnothing \neq \beta(U_1 \cap \ldots \cap U_n) \subseteq F(U_1) \cap \ldots \cap F(U_n) \subseteq \overline{F}(U_1) \cap \ldots \cap \overline{F}(U_n).$$

Thus $\{\overline{F}(U) : U \in \mathcal{U}\}$ is a family of closed subsets of the compact topological space $\mathbb{T}$, any finite subfamily of which has nonempty intersection. Hence it has non-empty intersection and (3.3) implies that $F \neq \varnothing$.

Thus there is $z \in \mathbb{T}$ such that $zx \in M_x^T$. By Lemma 3.1, then $z^2 x \in M_{zx}^T$. According to Lemma 3.2, the inclusions $zx \in M_x^T$ and $z^2 x \in M_{zx}^T$ imply $z^2 x \in M_x^T$. Proceeding inductively, we have $\{z^n x : n \in \mathbb{N}\} \subset M_x^T$. That is, $Q_z = \{z^n : n \in \mathbb{N}\} \subseteq F$. Since F is closed $\overline{Q_z} \subseteq F$. Since $1 \in \overline{Q_z}$ for any $z \in \mathbb{T}$, we see that $1 \in F$. That is, $x \in M_x^T$. $\qquad\square$

Lemma 3.4. *If $x \in X$ is such that $x \in M_x^T$, then $M_x^T = F_x^T$, where $F_x^T = \overline{\{T^n x : n \in \mathbb{Z}_+\}}$.*

Proof. Clearly $M_x^T \subseteq F_x^T$. Since $x \in M_x^T$ and $T(M_x^T) \subseteq M_x^T$, we see that $\{T^n x : n \in \mathbb{Z}_+\} \subseteq M_x^T$. Since M_x^T is closed, we have $F_x^T \subseteq M_x^T$. Thus, $F_x^T = M_x^T$. $\qquad\square$

Lemma 3.5. *If $x \in X$ is such that M_x^T has non-empty interior, then there exists $y \in X$ such that $y \in M_y^T = M_x^T$.*

Proof. Since M_x^T has non-empty interior, there is $k \in \mathbb{Z}_+$ such that $T^k x$ is an interior point of M_x^T. In particular, $y = T^k x \in M_x^T$. By Lemma 3.1, $M_y^T = M_{T^k x}^T = M_x^T$. Thus $y \in M_x^T = M_y^T$. $\qquad\square$

3.1 Rotations

Proposition 3.6. *Let X be a complex topological vector space, $z \in \mathbb{T}$ and $T \in L(X)$. Then T is $\mathfrak{U}$-frequently hypercyclic if and only if zT is $\mathfrak{U}$-frequently hypercyclic and $\mathfrak{U}(T) = \mathfrak{U}(zT)$.*

Proof. Let $x \in \mathfrak{U}(T)$. Then $M_x^T = N_x^T = X$, where the sets M_x^T and N_x^T are defined in (3.1) and (3.2). On the other hand, clearly $N_x^{zT} = N_x^T$. Hence $N_x^{zT} = X$. In particular, $x \in N_x^{zT}$. By Lemma 3.3, $x \in M_x^{zT}$. Then by Lemma 3.4, $M_x^{zT} = F_x^{zT}$, where F_x^{zT} is the closure of $\{(zT)^n x : n \in \mathbb{Z}_+\}$. Since x is a hypercyclic vector for T, Theorem B implies that x is a hypercyclic vector for zT and therefore $F_x^{zT} = X$. Hence $M_x^{zT} = X$ and $x \in \mathfrak{U}(zT)$. Thus $\mathfrak{U}(T) \subseteq \mathfrak{U}(zT)$. Applying this inclusion to T replaced by zT and z replaced by z^{-1}, we obtain $\mathfrak{U}(zT) \subseteq \mathfrak{U}(T)$. Thus $\mathfrak{U}(T) = \mathfrak{U}(zT)$. $\qquad\square$

3.2 Powers

Proposition 3.7. *Let X be a topological vector space and $T \in L(X)$. If $x \in X$ is such that the set M_x^T, defined in (3.1), has non-empty interior, then $x \in \mathfrak{U}(T)$.*

Proof. By Lemma 3.5, we can pick $y \in X$ such that $y \in M_y^T = M_x^T$. By Lemma 3.4, $M_y^T = F_y^T$, where F_y^T is the closure of $\{T^n y : n \in \mathbb{Z}_+\}$. Thus, F_y^T has non-empty interior and Theorem A implies that $F_y^T = X$. Since $F_y^T = M_y^T = M_x^T$, we have $M_x^T = X$. That is, $x \in \mathfrak{U}(T)$. $\qquad\square$

Corollary 3.8. *Let X be a topological vector space, $T \in L(X)$ be an $\mathfrak{U}$-frequently hypercyclic operator and $n \in \mathbb{N}$. Then T^n is also $\mathfrak{U}$-frequently hypercyclic. Moreover $\mathfrak{U}(T) = \mathfrak{U}(T^n)$.*

Proof. Let $x \in \mathfrak{U}(T)$. Then the set M_x^T defined in (3.1) coincides with X. Since the union of finitely many subsets of $\mathbb{Z}_+$ has positive upper density if and only if at least one of them has positive upper density, one can easily see that

$$X = M_x^T = \bigcup_{j=0}^{n-1} M_{T^j x}^{T^n}.$$

By Lemma 3.1, the sets $M_{T^j x}^{T^n}$ are closed. Hence there is $j \in \{0, \dots, n-1\}$ such that $M_{T^j x}^{T^n}$ has non-empty interior. By Proposition 3.7, $M_{T^j x}^{T^n} = X$. That is, $T^j x \in \mathfrak{U}(T^n)$. Exactly as for usual hypercyclicity, one can easily verify that if y is an $\mathfrak{U}$-frequently hypercyclic vector for $S \in L(X)$ and $R \in L(X)$ has dense range and commutes with S, then $Ry \in \mathfrak{U}(S)$. Applying this observation with $S = T^n$, $y = T^j x$ and $R = T^{n-j}$, we see that $T^n x \in \mathfrak{U}(T^n)$. Since x and $T^n x$ have the same orbits with respect to T^n (up to one element added or removed), we obtain $x \in \mathfrak{U}(T^n)$. Thus $\mathfrak{U}(T) \subseteq \mathfrak{U}(T^n)$. The inclusion $\mathfrak{U}(T^n) \subseteq \mathfrak{U}(T)$ is obvious. $\qquad\square$

Corollary 3.9. *Let X be a real topological vector space and $T \in L(X)$. Then T is $\mathfrak{U}$-frequently hypercyclic if and only if $-T$ is $\mathfrak{U}$-frequently hypercyclic and $\mathfrak{U}(T) = \mathfrak{U}(-T)$.*

Proof. By Corollary 3.8, $\mathfrak{U}(T) = \mathfrak{U}(T^2) = \mathfrak{U}((-T)^2) = \mathfrak{U}(-T)$. $\qquad\square$

4 Proof of Theorems 1.3 and 1.2

The above results allow us to strengthen Theorem 1.1.

Corollary 4.1. *Let X be a separable infinite dimensional Banach space, $z \in \mathbb{K}$ and $T \in L(X)$ be such that $|z| = 1$ and there exists $f \in X^* \setminus \{0\}$ for which $\lim_{n \to \infty} \|T^{*n} f\|^{1/n} = 0$. Then $zI + T$ is not $\mathfrak{U}$-frequently hypercyclic.*

Proof. By Corollary 2.2, $I + z^{-1}T$ is not $\mathfrak{U}$-frequently hypercyclic. Using Proposition 3.6 in the case $\mathbb{K} = \mathbb{C}$ and Corollary 3.9 in the case $\mathbb{K} = \mathbb{R}$, we see that $z(I + z^{-1}T) = zI + T$ is not $\mathfrak{U}$-frequently hypercyclic. $\qquad\square$

Since $\|T^{*n}f\| \leqslant \|f\|\|T^{*n}\| = \|f\|\|T^n\|$ for any $f \in X^*$ and any $n \in \mathbb{Z}_+$, we immediately obtain the following slightly stronger form of Theorem 1.1.

Theorem 4.2. *Let T be a bounded linear operator on a separable infinite dimensional Banach space X such that $\lim_{n \to \infty} \|T^n\|^{1/n} = 0$ and $z \in \mathbb{K}$ be such that $|z| = 1$. Then $zI + T$ is not $\mathfrak{U}$-frequently hypercyclic. In particular, $zI + T$ is not frequently hypercyclic.*

In order to prove Theorems 1.3 and 1.2, we need the following lemmas.

Lemma 4.3. *Let $X \in \Phi_{\mathbb{K}}$ and $T \in L(X)$. Assume also that T has no non-trivial closed invariant subspaces of finite codimension. Then $T = zI + S$, where $z \in \mathbb{K}$ and $\lim_{n \to \infty} \|S^n\|^{1/n} = 0$.*

Proof. Since $X \in \Phi_{\mathbb{K}}$, there are $z \in \mathbb{K}$ and $S \in \mathcal{S}(X)$ such that $T = zI + S$. Let $X_{\mathbb{C}} = X$ and $R_{\mathbb{C}} = R$ for any $R \in L(X)$ if $\mathbb{K} = \mathbb{C}$ and $X_{\mathbb{C}}$ be the complexification of X and $R_{\mathbb{C}} \in L(X_{\mathbb{C}})$ be the complexification (=the unique complex linear extension) of $R \in L(X)$ if $\mathbb{K} = \mathbb{R}$.

In any case $T_{\mathbb{C}} = zI + S_{\mathbb{C}}$ and $S_{\mathbb{C}} \in \mathcal{S}(X)$. The fact that T has no non-trivial closed linear subspaces of finite codimension is clearly equivalent to the statement that the point spectrum $\sigma_p(T_{\mathbb{C}}^*)$ of the dual $T_{\mathbb{C}}^*$ of $T_{\mathbb{C}}$ is empty.

Assume that $w \in \sigma(T_{\mathbb{C}}) \setminus \{z\}$. Then $w - z \in \sigma(S_{\mathbb{C}}) \setminus \{0\}$. Since a non-zero element of the spectrum of any strictly singular operator on a complex Banach space is a normal eigenvalue [24], we see that $w - z$ is a normal eigenvalue of $S_{\mathbb{C}}$ and therefore $w - z \in \sigma_p(S_{\mathbb{C}}^*)$. Hence $w \in \sigma_p(T_{\mathbb{C}}^*)$, which contradicts the equality $\sigma_p(T_{\mathbb{C}}^*) = \varnothing$. Thus $\sigma(T_{\mathbb{C}}) = \{z\}$. Hence $\sigma(S_{\mathbb{C}}) = \{0\}$. By the spectral radius formula $\lim_{n \to \infty} \|S_{\mathbb{C}}^n\|^{1/n} = 0$. Since S is the restriction of $S_{\mathbb{C}}$ to an $\mathbb{R}$-linear invariant subspace, $\|S^n\| \leqslant \|S_{\mathbb{C}}^n\|$ for any $n \in \mathbb{N}$ and therefore $\lim_{n \to \infty} \|S^n\|^{1/n} = 0$. $\qquad\square$

Lemma 4.4. *Let $X \in \Phi_{\mathbb{K}}$ and $T \in L(X)$ be a hypercyclic operator. Then $T = zI + S$, where $z \in \mathbb{K}$, $|z| = 1$ and $\lim_{n \to \infty} \|S^n\|^{1/n} = 0$.*

Proof. As it is shown in [26], a hypercyclic operator on a real or complex topological vector space has no non-trivial closed invariant subspaces of finite codimension. Hence, by Lemma 4.3, $T = zI + S$, where $z \in \mathbb{K}$ and $\lim_{n \to \infty} \|S^n\|^{1/n} = 0$. It remains to demonstrate that $|z| = 1$. Using the equalities $T = zI + S$ and $\lim_{n \to \infty} \|S^n\|^{1/n} = 0$, one can easily verify that

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = |z| \quad \text{for any } x \in X \setminus \{0\}. \tag{4.1}$$

Now if $|z| > 1$, then according to (4.1), $\|T^n x\| \to \infty$ as $n \to \infty$ for each non-zero $x \in X$. Similarly, if $|z| < 1$, then $\|T^n x\| \to 0$ as $n \to \infty$ for any $x \in X$. Hence T can not be hypercyclic if $|z| \neq 1$. Thus $|z| = 1$. $\square$

Theorem 1.3 follows immediately from Lemma 4.4 and Theorem 4.2.

Proof of Theorem 1.2. Let T be an $\mathfrak{u}$-frequently hypercyclic operator on a complex Banach space X. Assume also that z is an isolated point of the spectrum $\sigma(T)$. Let $x \in \mathfrak{u}(T)$, P be the spectral projection corresponding to the component $\{z\}$ of the spectrum of T, $Y = P(X)$ and $S \in L(Y)$ be the restriction of T to the closed invariant subspace Y. Clearly $\sigma(S) = \{z\}$. Since $S^n Px = PT^n x$ for each $n \in \mathbb{Z}_+$, we see that $Px \in \mathfrak{u}(S)$. Thus, S is an $\mathfrak{u}$-frequently hypercyclic operator on Y satisfying $\sigma(S) = \{z\}$. If $|z| < 1$, then $\|S^n y\| \to 0$, while $\|S^n y\| \to \infty$ if $|z| > 1$ for any $y \in Y$ and S can not be hypercyclic. Hence $z \in \mathbb{T}$. Since $\sigma(S) = \{z\}$, we have $S = zI + R$, where $\lim_{n \to \infty} \|R^n\|^{1/n} = 0$. We have obtained a contradiction with Theorem 4.2. $\square$

5 Concluding remarks

1. In the proofs of Proposition 3.6 and Corollary 3.8, we use the following property of the family Λ of subsets of $\mathbb{Z}_+$ of positive upper density. Namely, if a finite union of sets belongs to Λ, then one of them does. The family Λ_0 of subsets of $\mathbb{Z}_+$ of positive lower density fails to have this property. Thus our proof of stability of the class of $\mathfrak{u}$-frequently hypercyclic operators under powers and under multiplication by unimodular scalars does not work for the class of frequently hypercyclic operators. This leads to the following question.

Question 2. Let T be a frequently hypercyclic operator on a separable infinite dimensional Banach space X. Is it true that the operators T^n and zT for $n \in \mathbb{N}$ and $z \in \mathbb{K}$ with $|z| = 1$ are frequently hypercyclic?

2. Proposition 3.6, Corollary 3.8 and Corollary 3.9 admit the following generalization. Let Ω be any family of non-empty subsets of $\mathbb{Z}_+$ satisfying the following conditions

if $A \in \Omega$ and $A \subseteq B \subseteq \mathbb{Z}_+$, then $B \in \Omega$;

if $A \in \Omega$ and $k \in \mathbb{Z}_+$, then $A + k \in \Omega$ and $(A - k) \cap \mathbb{Z}_+ \in \Omega$;

if $k \in \mathbb{N}$ and A_j for $1 \leqslant j \leqslant k$ are subsets of $\mathbb{Z}_+$ such that $\bigcup_{j=1}^{k} A_j \in \Omega$, then there exists $j \in \{1, \dots, k\}$ such that $A_j \in \Omega$.

We say that x is an Ω-*hypercyclic* vector for a continuous linear operator T on a topological vector space X if $\{n \in \mathbb{Z}_+ : T^n x \in U\} \in \Omega$ for each non-empty open subset U of X and that T is Ω-*hypercyclic* if it has an Ω-hypercyclic vector.

All proofs in Section 3 work without any changes (just replace Λ by Ω) for Ω-hypercyclicity instead of $\mathfrak{u}$-frequent hypercyclicity. Thus, the class of Ω-hypercyclic operators is stable under taking the powers and under multiplying by unimodular scalars. Note also that when Ω is the set of all infinite subsets of $\mathbb{Z}_+$, then Ω-hypercyclicity coincides with the usual hypercyclicity. The following example seems to be interesting. Let Ψ be the set of all decreasing sequences

$\varphi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that $\sum\limits_{n=0}^{\infty} \varphi_n = \infty$. Clearly, for each $\varphi \in \Psi$, the family Ω_φ of subsets A of $\mathbb{Z}_+$ for which $\sum\limits_{n\in A} \varphi_n = \infty$ satisfies all properties from the above display. It is also easy to verify that an operator T is $\mathfrak{U}$-frequently hypercyclic if and only if it is Ω_φ-hypercyclic for any $\varphi \in \Psi$. This naturally leads to the following question.

Question 3. For which $\varphi \in \Psi$, there is an Ω_φ-hypercyclic operator on any separable infinite dimensional Banach space?

3. One can observe that all operators T *proven* to be frequently hypercyclic have a stronger property. Namely they admit a vector x such that for any non-empty open set U the set $\{n \in \mathbb{Z}_+ : T^n x \in U\}$ contains a subset, which has positive density. Since there are sets $A \subset \mathbb{Z}_+$ of positive lower density which contain no subsets of positive density, the last property of x is, formally speaking, stronger than to be a frequently hypercyclic vector. It might be interesting to find out whether all frequently hypercyclic operators admit vectors having this stronger property.

4. Herrero [20] have characterized the operator norm closure $\overline{\mathtt{HC}(\mathcal{H})}$ of the set $\mathtt{HC}(\mathcal{H})$ of hypercyclic operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$. Later Herrero and Wang [21] have shown that any $T \in \overline{\mathtt{HC}(\mathcal{H})}$ is a limit of a sequence T_n of elements of $\mathtt{HC}(\mathcal{H})$ such that the differences $T - T_n$ are compact. Using these results, one can easily prove the following proposition.

Proposition 5.1. *A nonempty compact subset K of $\mathbb{C}$ is the spectrum of some bounded hypercyclic operator T on a separable infinite dimensional complex Hilbert space $\mathcal{H}$ if and only if $K \cup \mathbb{T}$ is connected.*

Proof. The 'only if' part is proved in [20]. Let K be a non-empty compact subset of $\mathbb{C}$ such that $K \cup \mathbb{T}$ is connected. We have to show that there is a hypercyclic operator $T \in L(\mathcal{H})$ such that $\sigma(T) = K$. Let A be the set of isolated points of K. Exactly as in the proof of Theorem 1.2, one can easily see that $A \subset \mathbb{T}$. Clearly A is at most countable. Let $S \in L(\ell_2)$ be a compact weighted backward shift: $Se_0 = 0$ and $Se_n = w_n e_{n-1}$ for $n \in \mathbb{N}$, where $w = \{w_n\}_{n\in\mathbb{N}}$ is a sequence of non-zero complex numbers tending to 0 as $n \to \infty$. Consider the ℓ_2-direct sum T_0 of $zI + S$ for $z \in A$:
$$T_0 = \bigoplus_{z\in A}(zI + S).$$

Then $\sigma(T_0) = \overline{A}$. From [16][Theorem 5.1] if follows that T_0 is topologically mixing. Next, we take a bounded normal operator S_1 on a separable complex Hilbert space $\mathcal{H}_1$ such that $\sigma(S_1) = K \setminus A$ and S_1 has empty point spectrum. According to the mentioned results of Herrero [20] and Herrero and Wang [21], we have that $S_1 \in \overline{\mathtt{HC}(\mathcal{H}_1)}$ and therefore there exists a compact operator $K \in L(\mathcal{H}_1)$ such that $T_1 = S_1 + K$ is hypercyclic. For each $z \in \mathbb{C} \setminus \sigma(S_1) = \mathbb{C} \setminus (K \setminus A)$, the operator $S_1 - zI$ is invertible and therefore by the Fredholm theorem, either $T_1 - zI$ is invertible or $T_1^* - zI$ has non-trivial kernel (here T_1^* is the dual operator, not the Hilbert space adjoint). The latter case is not possible since the dual of any hypercyclic operator has empty point spectrum. Thus $\sigma(T_1) \subseteq \sigma(S_1)$. One the other hand, S_1 has purely continuous spectrum and T_1 is a compact perturbation of S_1. Hence $\sigma(S_1) \subseteq \sigma(T_1)$. Thus $\sigma(T_1) = K \setminus A$. Let $T = T_0 \oplus T_1$. Then $\sigma(T) = \sigma(T_0) \cup \sigma(T_1) = \overline{A} \cup (K \setminus A) = K$. Moreover T is hypercyclic as a direct sum of a hypercyclic operator with a mixing operator. $\square$

Question 4. What compact subsets of $\mathbb{C}$ are the spectra of frequently hypercyclic operators on ℓ_2? What about $\mathfrak{U}$-frequently hypercyclic operators?

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